NUMERICAL CALCULATION OF THE V(l) CURVES

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Abstract
Aim of this paper is to present results of recent research at FEE&IT Slovak University of Technology in the field of lighting theory. Research has been oriented to find an empirical formula for calculation of V (λ) and V´ (λ) curves by means of non-linear regression, when matching criterion of empirical to theoretical curve is the least-square method. In the paper, regression algorithm is lined out, followed by compilation of system of normal equations. Solution is statistically evaluated.

Introduction

All the photometric (luminous) quantities used in lighting engineering are, in fact, derived from radiant quantities by means of the function of relative spectral sensitivity of human eye, also known as the V (λ) curve. As early as 1924, the Comission Internationale de L'Eclairage (CIE) laid down a standard spectral eye sensitivity curve for photopic vision V (λ), followed in 1951 by a similar curve for scotopic vision. The curves give the relative photopic and scotopic eye sensitivity as a function of the wavelength. Meaning of those curves for lighting engineering is very important, however, recently yet tabelized values are used in practice. More convenient is to use a formula, at least for calculations when not high precision results are required.

Gaussian curves

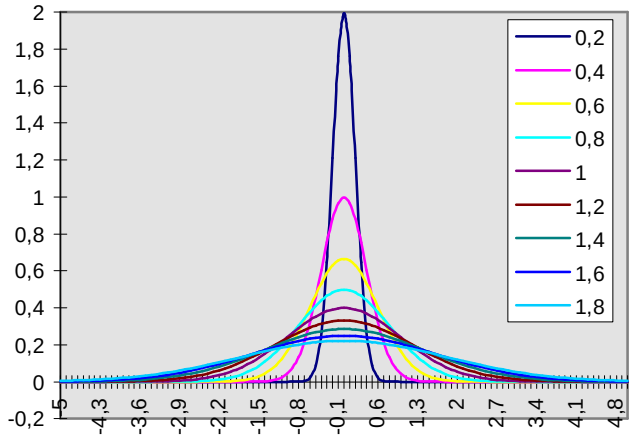

Fig. 1 Gaussian curves for μ = 0, σ varies

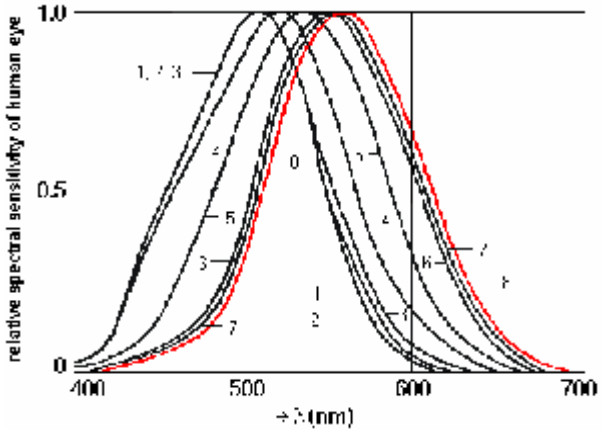

Fig. 2 Curves of spectral sensitivity of human eye for different luminance levels

1 - $L_a = 10^{-5}$ cd.m^{-2} (V′(λ) curve for scotopic vision)
2 - $L_a = 10^{-4}$ cd.m^{-2}
3 - $L_a = 10^{-3}$ cd.m^{-2}
4 - $L_a = 10^{-2}$ cd.m^{-2}
5 - $L_a = 10^{-1}$ cd.m^{-2}
6 - $L_a = 1$ cd.m^{-2}
7 - $L_a = 10$ cd.m^{-2}
8 - $L_a = 100$ cd.m^{-2} (V(λ) curve for fotopic vision)

Gaussian curves are, in fact, being a geometrical representation of the normal distribution function N (μ, σ) with two independent variables (parameters), which are well known from the statistics. Meaning of the parameters is apparent from fig. 1 and is as follows: μ - average of statistical variable

and σ - dispersion about average value. The function of normal distribution can be written in a form of equation:

$$N(m,s) = \frac{1}{s\sqrt{2p}} e^{-\frac{(x-m)^2}{2s^2}} \qquad (1)$$

where x - is a statistical variable

Theoretical background on regression

In plane Oxy let us have a given set of points $\{[x_i, y_i]: x_i, y_i \in R, i = 1, 2, ..., n\}$, where x_i and y_i are coordinates of a point lying on the V (λ) curve - (thus, to simplify notation $y \equiv V$ a $x \equiv \lambda$). Let us suppose that between quantities x and y a functional relation $y = f(x)$ is valid , where f (x) is function of a known form. In general theory of regression, it is further usually supposed that points $[x_i, y_i]$ are dispersed along curve $y = f(x)$ due to influence of measurement errors. In our case this kind of dispersion can be neglected supposing that V (λ) curves are standardized and therefore not loaded by errors (however, curves have initially been measured on very numerous and various sample of people).

Function f(x) contains generaly q ($q < n$) unknown constants, i.e. parameters, which can be designated $b_1, b_2,..., b_q$, so we can write $f(x) = f(x; b_1, b_2,..., b_q)$. Hence, functionality y_i on x_i can be expressed by so called *operative equation*

$y_i = f(x_i; b_1, b_2,..., b_q)$ $(i = 1, 2, ..., n)$ (2)

When to overlap a curve described by equation $y = f(x; b_1, b_2,..., b_q)$ throughover points $[x_1, y_1]$, $[x_2, y_2]$, ..., $[x_n, y_n]$, we need to statistically estimate unknown parameters $b_1, b_2,..., b_q$, which act in the equation. At the same time we require that the curve is to be overlaped as close as possible to points $[x_i, y_i]$. Estimation of parameters b_k is designated as b_k^*. The way how to determine b_k^* depends on chosen criterion saying how close is the curve to empirical points. In least-sqare method, sum of squares of differences $y_i = f(x_i; b_1^*, b_2^*,..., b_q^*)$ is used as the criterion, estimations $b_1^*, b_2^*,..., b_q^*$ are then determined as the quantities minimalizing that sum. Thus, when we denote

$$S = \sum_{i=1}^{n} \left[y_i - f(x_i; b_1^*, b_2^*, ..., b_q^*) \right]^2 , \qquad (3)$$

then estimations b_k^* $(k = 1, 2, ..., q)$ will be determined from condition S = min. This condition express the principle of the least-square method. Function $y = f(x; b_1, b_2,..., b_q)$ is also called *theoretical regression function* of variable y on variable x and its grafical interpretation is *theoretical regression curve*. Regression function in which unknown parameters b are replaced by their estimations is called *empirical regression function* $y_i = f(x_i; b_1^*, b_2^*,..., b_q^*)$ and its grafical interpretation - *empirical regression curve*. For $x = x_i$ and coresponding dependant variable y_i is from empirical curve possible to read the value $y_i^* = f(x_i; b_1^*, b_2^*,..., b_q^*)$. Difference $u_i = yi - y_i^*$ is called residue of variable y_i. Absolute value of that residue $|u_i|$ gives distance of point $[x_i, y_i]$ from point $[x_i, y_i^*]$ lying on overlaped curve. *Residual (error) sum of squares* is then

$$S_0 = \sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{n} \left[y_i - f(x_i; b_1^*, b_2^*, ..., b_q^*) \right]^2 \qquad (4)$$

and is used for estimation of dispersion of points $[x_i, y_i]$ along theoretical regression curve.

As the function S is a quadratic polynome **q** of variables $b_1^*, b_2^*,..., b_q^*$, we minimize it by common method of searching extremes of function of multiple variables, i.e. partial derivations of that function by individual variables are to be reset (= 0). Thus, we obtain a system of linear equations **q** with **q** unknowns - *system of normal equations*.

Solution of regression for Gaussian curves

It is generally known that the V (λ) curve is similar to Gaussian curve. How much similar? From tables for V (λ) defined in CIE publication and accepted in national standard STN 01 1710 we can observate that V (λ) curves are not symmetrical to axis passing trough the maximum of those curves

so differences between V (λ) and Gaussian curves will necessarily negatively impact to calculation accuracy. To match Gaussian curves now we set the V (λ) function to the form of equation (1)

$$V(\lambda) = \frac{p}{\sigma\sqrt{2\pi}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} \qquad (5)$$

where in role of statistical variable is now the wavelength λ of optical radiation and **p** is a parameter expreessing dilatation of curve along the x-axis. Average value of curve described by equation (5) we determine directly from V (λ) curves - μ = 555 for photopic vision V (λ) and μ = 507 for scotopic vision V′ (λ). Hence, expression (5) we will find as unknown function of two parameters V (p, σ).

Equation (5) is non-linear function of parameter s. Now we develop the Gaussian function to Taylor's series. First, from empirical values $[x_i, y_i]$, approximate values $p^{(0)}$ and $\sigma^{(0)}$ for parameters of function V = f (λ; p, σ) are to be determined. Then the function V = f (λ; p, σ) is to be developed to Taylor's series in surroundings of the point $[p^{(0)}, \sigma^{(0)}]$ and neglected members of the second and higher orders:

$$V = f_0 + f_p(p - p^{(0)}) + f_\sigma(\sigma - \sigma^{(0)}) \qquad (6)$$

where

$$f_0 = f(\lambda; p^{(0)}, \sigma^{(0)}) = \frac{p^{(0)}}{\sigma^{(0)}\sqrt{2\pi}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}} \qquad (7)$$

$$f_p = \left.\frac{\partial f(\lambda; p, \sigma)}{\partial p}\right|_{[p,\sigma]=[p^{(0)},\sigma^{(0)}]} = \frac{1}{\sigma^{(0)}\sqrt{2\pi}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^{(0)2}}} \qquad (8)$$

$$f_\sigma = \left.\frac{\partial f(\lambda; p, \sigma)}{\partial \sigma}\right|_{[p,\sigma]=[p^{(0)},\sigma^{(0)}]} = p^{(0)} \frac{(\lambda-\mu)^2 - \sigma^{(0)2}}{\sigma^{(0)4}\sqrt{2\pi}} e^{-\frac{(\lambda-\mu)^2}{2\sigma^{(0)2}}} \qquad (9)$$

Least-square condition is applied to function (6) which is non-linear for unknowns $\Delta_p^{(0)} = p - p^{(0)}$ and $\Delta_\sigma^{(0)} = \sigma - \sigma^{(0)}$. Solving the system of normal equations we obtain values $\Delta_p^{(1)}$ and $\Delta_\sigma^{(1)}$. From these we calculate the next approximation of estimated parameters $p^{(1)} = p^{(0)} + \Delta_p^{(1)}$ and $\sigma^{(1)} = \sigma^{(0)} + \Delta_\sigma^{(1)}$. Procedure is to be repeated with those approximations.

But the member $\mathbf{p^{(0)} + D_p^{(0)} = p}$, therefore is better to search for function $\mathbf{V = f(\lambda; p, D_s)}$. Though it is natural, because dependance of Gaussian function on **p** is linear.

$$V = \varphi_1(\lambda).p + \varphi_2(\lambda).\Delta_\sigma \qquad (10)$$

Sum of squares then will be equal to

$$S = \sum_{i=1}^{n} \left[V_i - j_1(l_i).p - j_2(l_i).D_s \right]^2 \qquad (11)$$

Partial derivations of sum of squares for both parameters are

$$\frac{\partial S}{\partial p} = -2\left[\sum_{i=1}^{n} j_1(l_i)V_i - p\sum_{i=1}^{n} j_1^2(l_i) - D_s \sum_{i=1}^{n} j_1(l_i)j_2(l_i) \right] = 0$$

$$\frac{\partial S}{\partial D_s} = -2\left[\sum_{i=1}^{n} j_2(l_i)V_i - p\sum_{i=1}^{n} j_1(l_i)j_2(l_i) - D_s \sum_{i=1}^{n} j_2^2(l_i) \right] = 0$$

and system of normal equations is resulting to

$$p\sum_{i=1}^{n} j_1^2(l_i) + D_s \sum_{i=1}^{n} j_1(l_i) j_2(l_i) = \sum_{i=1}^{n} j_1(l_i) V_i$$

$$p\sum_{i=1}^{n} j_1(l_i) j_2(l_i) + D_s \sum_{i=1}^{n} j_2^2(l_i) = \sum_{i=1}^{n} j_2(l_i) V_i \qquad (12)$$

From the system of normal equations follow formulas for unknown parameters

$$p = \frac{\sum j_2^2 \sum j_1 V_i - \sum j_1 j_2 \sum j_2 V_i}{\sum j_2^2 \sum j_1^2 - \left(\sum j_1 j_2\right)^2} \qquad (13)$$

$$D_s = \frac{\sum j_2 V_i - a \sum j_1 j_2}{\sum j_2^2} \qquad (14)$$

Finally, following statistical parameters are evaluated:

1. **average quadratic error** e^2 - represents average value of errors $(V-V^*)^2$
2. **dispersion** s^2 - dispersion of errors $(V-V^*)^2$ around average value ε^2
3. **decisive abberation** s

$$e^2 = \frac{1}{n}\sum_i \left(V_i - V_i^*\right)^2 \qquad s^2 = \frac{1}{n}\sum_i \left((V_i - V_i^*)^2 - e^2\right)^2$$

Results

Two alternatives have been studied: approximation with solid and divided function. Due to non-symmetrical nature of the V (λ) curve it was shown by analysis that second alternative is due to higher precision preferred to be used, disadvantage is, however, in use of two different formulas according to range of wavelength λ whis has been divided to two semiranges. Empirical formulas derived by author of the paper are as follows.

l = **380 - 540 nm**		l = **540 - 780 nm**	
$V(l) = 1{,}032.e^{-\frac{(l-555)^2}{2{,}845.10^3}}$	(15a)	$V(l) = e^{-\frac{(l-555)^2}{4{,}219.10^3}}$	(15b)
l = **380 - 490 nm**		l = **490 - 780 nm**	
$V'(l) = 0{,}989.e^{-\frac{(l-507)^2}{3{,}728.10^3}}$	(16a)	$V'(l) = e^{-\frac{(l-507)^2}{2{,}526.10^3}}$	(16b)

So for each of both curves V (λ) and V' (λ) there exist two different formulas depending on the value of λ. Parameters of regression curves are shown in table 1, where $k_1 = \frac{p}{s\sqrt{2p}}$ and $k_2 = 2s^2$ simplify the formula for Gaussian curve.

Table 1 Parameters of regression curves V (λ) and V´ (λ)

curve V (l)	p	s	k_1	k_2
380 - 540 nm	97,54574	37,71606	1,031792	2845,002
540 - 780 nm	116,03780	45,93143	1,007858	4219,393

curve V´ (l)	p	s	k_1	k_2
380 - 490 nm	107,10180	43,17905	0,989541	3728,861
490 - 780 nm	89,08644	35,54185	0,999958	2526,446

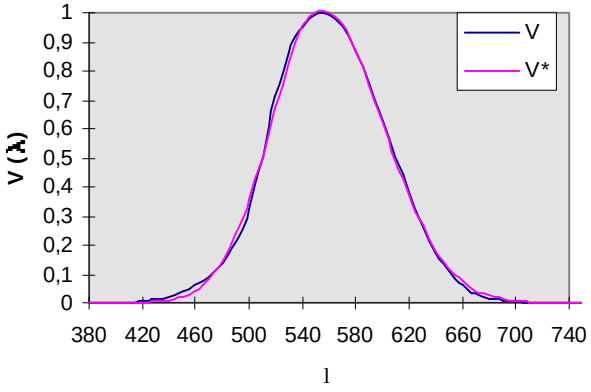

Fig. 3 Empirical and regression curves V (λ)

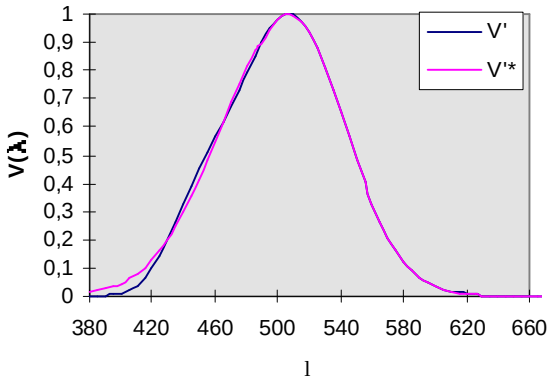

Fig. 4 Empirical and regression curves V´ (λ)

Discussion and conclusions

Statistical evaluation of regresion is shown on figures 5, 6 and in table 2. Maximum error do not exceed 4,5 % but average error is only about 1,4 % for the whole range both for V (λ) and V´ (λ) curves. Note that maximum error is only observable in very narrow part of spectrum.

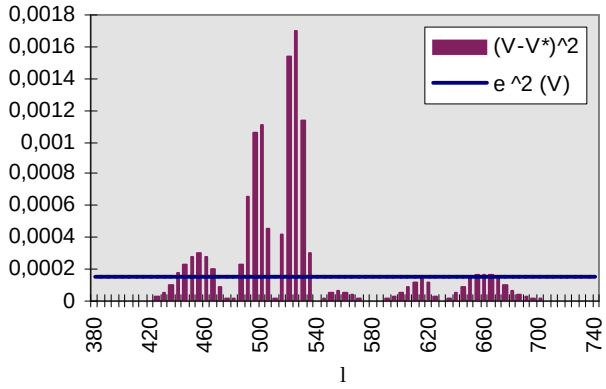

Fig. 5 Statistical eval. of regression for V (λ)

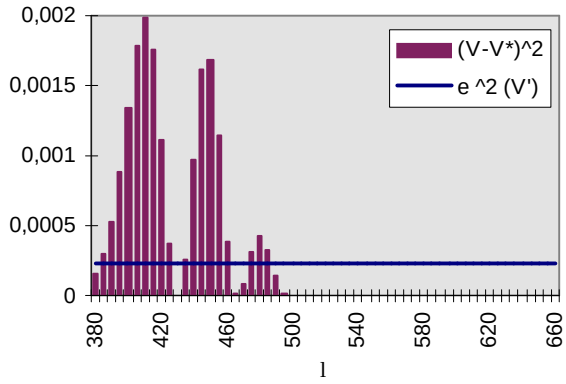

Fig. 6 Statistical eval. of regression for V´ (λ)

Table 2 Statistical evaluation of regression

	e^2	s^2	s	
curve V (l)	0,000156	$8,77.10^{-8}$	0,000296	
curve V´ (l)	0,000223	$1,99.10^{-7}$	0,000446	

Note: Statistical evaluation is common for the whole curve V (λ), i.e. using both formulas on whole range

As it can be seen from figure 6, no error is observable in right part of spectrum for V´ (λ)! But it is difficult to match any Gaussian curve to the very „coarse“ left part, indeed.

Algorithm of non-linear regression has been programmed in Borland C++ language. In different cases, different values for 0^{th} approximation have been used. Also being the estimation quite far from 0^{th} approximation, the calculation converged very fast ($i < 15$), but sometimes, at quite good estimations started the calculation diverge. Generally said, it was necessary to reach 4 up to 12 iterations for different cases.

Results of searching for empirical formulas are presented in a form of equations (15) to (16). Those are simple mathematical equations that with relative good precision match the original spectral sensitivity curves of human eye defined in standards. It is possible to use the formulas for different kinds of calculations in lighting engineering branch. However, if very high precision is required, i.e. to calibrate measuring devices, it is suitable to turn to more accurate values from tables or... We have prepared a new approach to increase the accuracy as much as possible and first results obtained recently show that this is possible. Results of further research will be presented soon, however, new formulas are more complicated and suitable mostly for computeral calculations. Formulas presented in this paper are suitable rather for fast manual calculations sometimes needed in practice of a lighting engineer.

References

[1] Slovak National Standard STN 01 1710

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