Balkan Light'99

Calculation of Illuminance Distribution Inside the Cylindrical Area Using Sphere Mathematical Models

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Abstract: Luminous flux and plane illuminance distributions on cylinder surfaces are determined in this paper. Interior of the cylinder is illuminated by a point light source (PLS), of known position and known luminous intensity distribution function (LIDF). Cylinder surfaces are of ideally diffuse characteristics (Lambertian surfaces). System of interreflection equations of luminous flux (SIE-LF) and mathematical model of the equivalent illuminated sphere, which was developed in the authors' previous papers, are used for solving this problem. This paper presents a progression in equivalent sphere model generalization in case of cylinder's nonuniform painted envelope.

1. INTRODUCTION*

Closed spaces in the shape of sphere and cylinder are unfrequent architectural-constructive instances, and a requirement for the plane illuminance distribution analysis in the inside of these spaces is rare. However, the uniform ideally diffuse painted sphere is a component part of the very important experimental system for luminous flux measurement and for reflective characteristics' materials measurement (Ulbricht sphere [1]).

Last several years, the first author has accomplished a detail characterization of the illuminated sphere in his papers, using the SIE-LF. It is supposed that the sphere is arbitrary, but ideally diffuse painted in the occasion of problem analysis ([2], [3], [4], [5]). All illumination parameters are obtained in the very simple manner, because the coefficients of interreflection (form factors), as the most complex problem for interior lighting calculation, in case of sphere geometry are calculated in the very simple manner.

In paper [4] several possibilities of the mathematical model of the illuminated sphere using in different purposes are demonstrated. Approximate lighting calculation of the closed space in the shape of cylinder, whose envelope is a uniform ideally diffuse surface, as well as the bases of cylinder, is one of the applications. In paper [5] authors have developed the exact equivalent sphere model with which the illuminated cut cone, and the cylinder from [4], are replaced, respectively.

The most complex problem of the illuminated cylinder is considered in this paper. The envelope of cylinder consists of two cylindrical parts. The equivalent mathematical sphere model which gives accurate solutions for the average plane illuminance distribution on the cylinder bases and on two surfaces which the envelope consists of, is developed in the paper. Very complex expressions for the form factors' cylindrical parts are also deduced in the paper. Besides two accurate models, the approximate one is presented also. Calculations are shown tabularly for a concrete example because of the comparison.

2. OUTLINE OF THE METHOD

2.1. Mathematical model of illuminated cylinder

The closed space in shape of cylinder whose all surfaces are ideally diffuse painted and of known reflectances is considered. Radius of cylinder's cross section is R, and the height is H. Bases' surfaces are $S_{c1} = \pi R^2$ (upper) and $S_{c4} = \pi R^2$ (lower), and their reflectances are ρ_{c1} and ρ_{c4}, respectively. Cylinder's envelope consists of two cylindrical parts $S_{c2} = \alpha R H$ and $S_{c3} = (2\pi - \alpha) R H$, where α is set polar angle. Reflectances of these surfaces are ρ_{c2} and ρ_{c3}, respectively. Total surface of the cylinder is $S_c = 2\pi R^2 + 2\pi R H$.

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* **This paper, among with several other papers, in which authors established the mathematical model of equivalent sphere, in realy meaning of the word, was writen in between two NATO bombing of the Ni{ town, in spring 1999.**

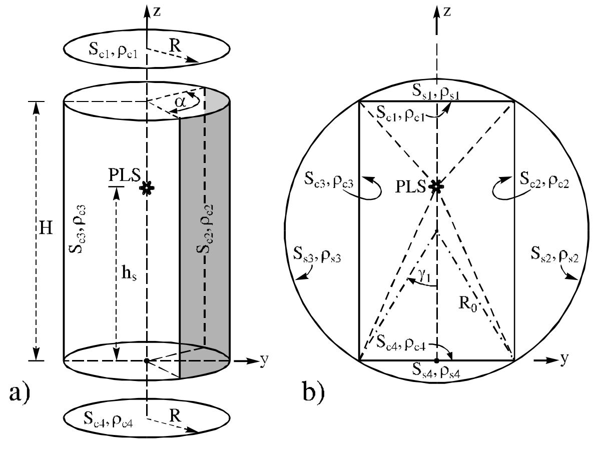

Fig. 1 Illuminated cylinder and equivalent sphere

Interior of the cylinder is illuminated by means of the PLS arbitrary placed on the cylinder z-axis, $z = h_s$. The LIDF of PLS is known, $I(\vartheta_k) = \; = I(\gamma, \varphi)$, so the direct luminous fluxes on cylinder's surfaces can be calculated on the basis of the definition expression

$$\Phi_{c0k} = \int_{\Omega_{ck}} I(\vartheta_k)\, \mathrm{d}\Omega_{ck}\,, \quad k = 1,2,3,4\,, \tag{1}$$

where Ω_{ck}: solid angle under which the k-th surface is viewed from the PLS.

The basis for this problem solving is the law of the luminous flux conservation which is expressed by the SIE-LF of the form ([6], [7], [8], [2], [3]):

$$\Phi_{c0k} = \sum_{i=1}^{4} (\delta_{ik} - \rho_{ci}\; f_{ci,ck})\, \Phi_{ci}\;,\; k = 1,2,3,4\,, \tag{2}$$

where δ_{ik}: Kronecker symbol, $\Phi_{ci} = \Phi_{c0i} + \Phi_{ind\,ci}$: total luminous flux on the i-th surface (Φ_{c0i} - direct and $\Phi_{ind\,ci}$ - indirect component) and $f_{ci,ck}$: form factor among the i-th and the k-th cylinder surfaces, whose general form is

$$f_{ci,ck} = \frac{1}{S_{ci}} \iint_{S_{ci}\,S_{ck}} \frac{\cos\gamma_{ci}\;\cos\gamma_{ck}}{\pi\, r_{ik}^2}\, \mathrm{d}S_{ci}\, \mathrm{d}S_{ck}\;,\; i,k = 1,2,3,4\,. \tag{3}$$

All form factors, except one for $i,k = 2,3$, can be, as in [4], calculated by means of the equivalent sphere which is obtained when the cylinder is spherumscribed. General relations are valid for the form factors

$$\sum_{k=1}^{4} f_{ci,ck} = 1\,,\; i = 1,2,3,4\,, \tag{4}$$

$$S_{ci}\;f_{ci,ck} = S_{ck}\;f_{ck,ci}\,. \tag{5}$$

Projections of the cylinder surfaces on the sphere are S_{si}, $i = 1,2,3,4$, and can be calculated according to Fig.1b and on the basis of the following expressions: $S_{s1} = 2\,\pi\, R_0^2\,(1 - \cos\gamma_1)$, $S_{s2} = 2\,\alpha\, R_0^2\,\cos\gamma_1$, $S_{s3} = 2\,(2\,\pi - \alpha)\, R_0^2\,\cos\gamma_1$, $S_{s4} = 2\,\pi\, R_0^2\,(1 - \cos\gamma_1)$. Total surface of the sphere is $S_s = 4\,\pi\, R_0^2$. Form factors $f_{ci,ck}$ are now calculated on the basis of expressions (6a)-(6c) which are given here without deducing:

$$f_{c1,c1} = f_{c4,c4} = 0\,, \tag{6a}$$

$$f_{ci,ck} = (S_{si} / S_{ci})\cdot(S_{sk} / S_s)\,,\;\; (i = 1,4\,,\; k = 2,3,4\,),\;\; (i = 2,3\,,\; k = 1,4\,) \tag{6b}$$

$$\begin{aligned} 2\pi\alpha\, f_{c2,c2} &= 2\pi\alpha + \alpha^2 X^{-1} - 8\left[Y \operatorname{arctg}(X^{-1}Y^{-1}) + 0.5\, X^{-1} \ln(1 + X^2 Y^2)\right] - \\ &- 4\,(X + X^{-1})\left\{\alpha (1 + X^2)^{-1/2} \operatorname{arctg}\left[\sqrt{1 + X^2}\, \operatorname{tg}(\alpha/2)\right] - 2 \int_{x=0}^{\alpha/2} \left[x / (1 + X^2 \sin^2 x)\right] dx\right\}, \end{aligned} \tag{6c}$$

where $X = 2R / H$, $Y = \sin(\alpha/2)$ and $\alpha \le 180^{\circ}$. Form factors $f_{c2,c3}$, $f_{c3,c2}$ and $f_{c3,c3}$ are calculated by (4), (5) and (6c).

Solutions for $f_{ci,ck}$ and Φ_{c0k}, $i,k = 1,2,3,4$, are changed in system (2) and then it is solved, the luminous flux distribution on the cylinder surfaces is obtained. Average values of the plane illuminance (E_{c0k} - direct and $E_{ind\,ck}$ - indirect component) are

$$E_{c0k} = \Phi_{c0k} / S_{ck}, \quad k = 1, 2, 3, 4, \tag{7a}$$

$$E_{ind\,ck} = \Phi_{ind\,ck} / S_{ck} = (\Phi_{ck} - \Phi_{c0k}) / S_{ck}, \quad k = 1, 2, 3, 4. \tag{7b}$$

Expressions (7a) and (7b) will be denoted as the standard model solutions, model (a).

2.2. Simplified sphere model

The sphere which also consists of four surfaces as well as the cylinder $S'_{si} = S_{ci}$, $i = 1, 2, 3, 4$, whose reflectances are respectively $\rho'_{si} = \rho_{ci}$, $i = 1, 2, 3, 4$, and its total surface is $S'_s = S_c$, is used for an approximate calculation of the plane illuminance distribution. Form factors for the sphere of such kind are calculated in the simple manner ([3], [4]), i.e. putting $f_{ci,ck} \cong f'_{si,sk} = S'_{sk} / S'_s = S_{ck} / S_c$ solutions for indirect components of the plane illuminance are obtained

$$E_{ind\,ck} \cong E'_{ind\,s} = \left(\sum_{i=1}^{4} \rho_{ci}\, \Phi_{c0i} \right) \Bigg/ \left(S_c - \sum_{i=1}^{4} \rho_{ci}\, S_{ci} \right), \quad k = 1, 2, 3, 4. \tag{8}$$

The simplified sphere model gives approximate solutions i.e. the uniform distribution of indirect component of the plane illuminance on the sphere, respectively on the cylinder. This model will be denoted as (b).

2.3. Equivalent sphere model

The equivalent sphere model presents a sphere which is spherumscribed around the cylinder and consists of surfaces given in paragraph 2.1. In order to this sphere, as a closed illuminated space, would be equivalent to the illuminated space of a cylinder it is necessary to determine equivalent reflectances of their surfaces. Their reflectances are denoted as ρ_{si}, $i = 1, 2, 3, 4$.

Equivalent sphere problem can be modeled in several ways. Here, because of limited paper space, one model which gives a solution in advance of required accuracy is represented.

In that purpose SIE-LF (2) is at first rearranged, as in [2], in the form

$$\Phi_{c0k} = \sum_{i=1}^{4} (\delta_{ik} / \rho_{ci} - f_{ci,ck}) S_{ci} M_{ci}, \quad k = 1, 2, 3, 4, \tag{9a}$$

where $M_{ci} = \rho_{ci} \Phi_{ci} / S_{ci}$: exitance of the *i*-th cylinder surface, $f_{ci,ck}$, $i, k = 1, 2, 3, 4$: form factors of the cylinder given by expressions (6a)-(6c). The similar SIE-LF can be set up for the equivalent sphere also

$$\Phi_{s0k} = \sum_{i=1}^{4} (\delta_{ik} / \rho_{si} - f_{si,sk}) S_{si} M_{si}, \quad k = 1, 2, 3, 4, \tag{9b}$$

where $\Phi_{s0k} = \Phi_{c0k}$: direct luminous flux on the *k*-th sphere surface, ρ_{si}: unknown reflectance of the *i*-th surface of the equivalent sphere, $M_{si} = \rho_{si} \Phi_{si} / S_{si}$: exitance of the *i*-th sphere surface, and $f_{si,sk} = S_{sk} / S_s$, $i, k = 1, 2, 3, 4$: form factors among the *i*-th and the *k*-th surfaces of the equivalent illuminated sphere.

It establishes on the basis of (6b) that the coefficients out of diagonals in (9a) and (9b), except for $i, k = 2, 3$, satisfy reciprocity law (5) and are the same,

$$f_{ci,ck}\, S_{ci} = f_{si,sk}\, S_{si}, \quad i, k \neq 2, 3, \quad i \neq k. \tag{10}$$

In order to establish equalization (9a) and (9b) it is necessary at first to make sure that the reciprocity (10) is valid for every $i, k = 1, 2, 3, 4$ and $i \neq k$. Authors developed several models (approximate and accurate) for solving this problem. Only one of them, denoted as the iterately model of the equivalent sphere is represented in this paper.

In that purpose it is necessary to rearrange the second and the third equation in (9a). It is done in the following way

$$\Phi_{c02} + \Delta\Phi = \mathbf{L} + (1/\rho_{c2} - f_{c2,c2} + \Delta f_{22}) S_{c2} M_{c2} - (f_{c3,c2} + \Delta f_{32}) S_{c3} M_{c3} - \mathbf{L} \qquad (11b)$$

$$\Phi_{c03} - \Delta\Phi = \mathbf{L} - (f_{c2,c3} + \Delta f_{23}) S_{c2} M_{c2} + (1/\rho_{c3} - f_{c3,c3} + \Delta f_{33}) S_{c3} M_{c3} - \mathbf{L} \qquad (11c)$$

where: $\Delta\Phi = \Delta f_{22} S_{c2} M_{c2} - \Delta f_{32} S_{c3} M_{c3}$. Values Δf_{32} and Δf_{23} are obtained requiring that coefficients in (11b) and (11c) are reciprocal and the same to the corresponding one in (9b), i.e.

$$(f_{c3,c2} + \Delta f_{32}) S_{c3} = f_{s3,s2} S_{s3}, \qquad (12a)$$

$$(\Delta f_{23} + f_{c2,c3}) S_{c2} = f_{s2,s3} S_{s2}, \qquad (12b)$$

and Δf_{22} and Δf_{33} on the basis of (4) and (12), i.e. it follows that $\Delta f_{22} = \Delta f_{23}$ and $\Delta f_{33} = \Delta f_{32}$. Finally it is obtained for $\Delta\Phi$ on a left side of the equations (11b) and (11c) that

$$\Delta\Phi = \Delta f_{23} S_{c2} (M_{c2} - M_{c3}) = \Delta f_{23} S_{c2} (M_{s2} - M_{s3}). \qquad (13)$$

It is necessary for equalization of the sphere and cylinder to determine the reflectances ρ_{sk}, $k = 1, 2, 3, 4$ from the diagonally elements of the rearranged system (11) and (9b). Expressions have got the same form as in [5], i.e.

$$\rho_{sk} = \rho_{ck} / [\rho_{ck} + (1-\rho_{ck}) S_{ck} / S_{sk}], \; k = 1, 2, 3, 4. \qquad (14)$$

In order to accomplish the complete equalization it is necessary to calculate independent elements in (11b) and (11c), i.e. to calculate $\Delta\Phi$ which is unknown. Consequently it approaches a iterately procedure which in the ***n***-th step gives

$$\Delta\Phi^{(n)} = \Delta f_{23} S_{c2} (M_{s2}^{(n-1)} - M_{s3}^{(n-1)}), \qquad (15)$$

and exitances $M_{sk}^{(n-1)}$, $k = 1, 2, 3, 4$, are calculated on the basis of the general expression ([2], [5])

$$M_{sk}^{(n-1)} = \rho_{sk} (\Phi_{s0k}^{(n-1)} / S_{sk} + E_{ind\,s}^{(n-1)}), \; k = 1, 2, 3, 4, \qquad (16)$$

where $\Phi_{s0k}^{(n-1)}$ and $E_{ind\,s}^{(n-1)}$ consider a widening on the left side of the equations (11b) and (11c), i.e. $\Phi_{s0k}^{(n-1)} = \Phi_{s0k}$, $k = 1, 4$, and $\Phi_{s0k}^{(n-1)} = \Phi_{s0k} + (-1)^k \Delta\Phi^{(n-1)}$, $k = 2, 3$, and according to [3] and [4]

$$E_{ind\,s}^{(n-1)} = \left(\sum_{i=1}^{4} \rho_{si} \Phi_{s0i}^{(n-1)} \right) \Big/ \left(S_s - \sum_{i=1}^{4} \rho_{si} S_{si} \right). \qquad (17)$$

It embraces for the starting solutions for $\Phi_{s0k}^{(0)}$, $E_{ind\,s}^{(0)}$, $\Phi_{s0k}^{(0)} = \Phi_{s0k}$, $k = 1, 2, 3, 4$, respectively. Procedure is stopped when

$$\left| (\Delta\Phi^{(n)} - \Delta\Phi^{(n-1)}) / \Delta\Phi^{(n)} \right| \le \varepsilon, \qquad (18)$$

where ε is in advance of required inaccuracy of the calculation.

When condition (18) is accomplished finally expressions for calculations of the luminous flux and indirect component of the plane illuminance distributions on cylinder surfaces are obtained

$$\Phi_{ck}^{(n)} = S_{ck} (\rho_{sk} / \rho_{ck}) (\Phi_{s0k}^{(n)} / S_{sk} + E_{ind\,s}^{(n)}), \; k = 1, 2, 3, 4, \qquad (19)$$

$$E_{ind\,ck} = \Phi_{ck}^{(n)} / S_{ck} - \Phi_{c0k} / S_{ck}, \; k = 1, 2, 3, 4. \qquad (20)$$

Numerical experiments show that, for inaccuracy less than 1% only one to two iterations are necessary. This model will be denoted as (c).

Notice: If the accurate solutions for $\Phi_{c2}^{(n)}$ and $\Phi_{c3}^{(n)}$ from (19) are used, then it can be calculated an accurate value for equivalent reflectance of an uniform painted envelope of the cylinder. Then the equivalent sphere model from [5] can be directly used. As a matter of fact, using an average instead the accurate value for envelope equivalent reflectance, what is often used in practical calculations, gives later an approximate solution for the luminous flux distribution (e.g. in [3] and [7]).

3. NUMERICAL RESULTS

The program package for numerical calculations on PC computer is made on the basis of the described theoretical procedure.

Numerical results for the cylinder of radius $R = 1\,\text{m}$ and of height $H = 2\,\text{m}$, from a great number of numerical experiments, are given in this paper. Reflectances of the cylinder surfaces are $\rho_{c1} = 0.8$, $\rho_{c2} = 0.1$, $\rho_{c3} = 0.6$ and $\rho_{c4} = 0.2$. Total installed luminous flux of the PLS is $\Phi_0 = 1000\,\text{lm}$. Results are given in the case of the isotropic PLS, $I(\gamma,\varphi) = I_0$, and with the realistic LIDF, $I(\gamma,\varphi) = I_0 \cos\gamma$, $\gamma \in [0, \pi/2]$. The inaccuracy is $\varepsilon \le 10^{-3}\,\%$.

In Table 1 the iterative procedure for the indirect component of the plane illuminance calculation is illustrated in the case of using the isotropic PLS, which is put at height $h_s = 0.8H$. Angle α of the cylindrical part is $\alpha = 180^{\mathbf{o}}$. Values which are obtained using the model (a) are given in the last column of the table, due to comparison.

Table 1

$\alpha = 180^{o}$, $h_s = 0.8H$			$E_{ind\ ck}$ [lx]						
k	ρ_{ck}	ρ_{sk}	n=0	n=1	n=2	n=3	n=4	n=5	(a)
1	0.8	0.824	26.489	27.955	28.111	28.127	28.129	28.129	28.129
2	0.1	0.136	48.905	45.230	44.840	44.799	44.794	44.794	44.794
3	0.6	0.680	32.744	39.033	39.701	39.772	39.779	39.780	39.780
4	0.2	0.227	41.599	43.211	43.382	43.401	43.403	43.403	43.403

Values of the indirect component of the plane illuminance distribution in case of the illuminated cylinder from Table 1, now only for angle $\alpha = 90^{\mathbf{o}}$ are presented in Table 2. It can be concluded from presented values that the proposed method gives very accurate results for luminous flux and plane illuminance distribution, and for inaccuracy less than 1% one to two iterations are necessary.

Table 2

$\alpha = 90^{\mathbf{o}}$, $h_s = 0.8H$			$E_{ind\ ck}$ [lx]		
k	ρ_{ck}	ρ_{sk}	(a)	(b)	(c)
1	0.8	0.824	43.759	57.116	43.758
2	0.1	0.136	59.762	57.116	59.762
3	0.6	0.680	55.043	57.116	55.043
4	0.2	0.227	60.588	57.116	60.588

Results for the indirect component of the plane illuminance on the cylinder surfaces when the cylinder is illuminated of the PLS with the realistic LIDF, $I(\gamma,\varphi) = I_0 \cos\gamma$, $\gamma \in [0, \pi/2]$, are presented in Tables 3 and 4. Total installed luminous flux of the light source is $\Phi_0 = 1000\,\text{lm}$. The PLS is put at height $h_s = H$. Results in Table 3 are referred to cylinder for $\alpha = 180^{\mathbf{o}}$, and in Table 4 when the angle $\alpha = 90^{\mathbf{o}}$.

Table 3

$\alpha = 180^{\mathbf{o}}$, $h_s = H$			$E_{ind\ ck}$ [lx]		
k	ρ_{ck}	ρ_{sk}	(a)	(b)	(c)
1	0.8	0.824	28.931	28.294	28.931
2	0.1	0.136	29.351	28.294	29.351
3	0.6	0.680	24.353	28.294	24.353
4	0.2	0.227	29.698	28.294	29.698

Table 4

$\alpha = 90^{\mathbf{o}}$, $h_s = H$			$E_{ind\ ck}$ [lx]		
k	ρ_{ck}	ρ_{sk}	(a)	(b)	(c)
1	0.8	0.824	44.510	43.126	44.510
2	0.1	0.136	44.272	43.126	44.272
3	0.6	0.680	39.568	43.126	39.568
4	0.2	0.227	46.828	43.126	46.828

4. CONCLUSION

One new method for calculation of the indirect component of the plane illuminance distribution on the illuminated cylinder surfaces is presented in this paper. This method is based on a method of the equivalent illuminated sphere which the authors proposed in their previous' papers ([2], [3], [4], [5]).

Results obtained by the proposed method are characterized with a high accuracy as well as the standard method (a), which is based on the SIE-LF. Unlike this, it can be avoided by the proposed method the form factors determination, i.e. double integration on surfaces, because the form factors in case of equivalent sphere are very simple calculated.

Developed method has a practical importance for the accomplishment of a calculation of the luminous flux and plane illuminance on the illuminated surfaces although in unfrequented architectural-constructive instances.

Another practical importance of this paper refers to the measurement systems in a lighting engineering which use a sphere as a component part (Ulbricht sphere), which is very difficult to make. Equalization of two illuminated spaces would make possible the replacement of the sphere by e.g. the cylinder, or an another closed space of equivalent illumination characteristics as well as the sphere. In the meantime, authors developed the equivalent sphere model which does not require iterative procedure. These researches are in progress.

ACKNOWLEDGMENT: This paper was partly supported by the MST of the Republic of Serbia.

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